SEEMOUS 2013 PROBLEMS AND SOLUTIONS

Problem 1

Find all continuous functions $f : [1, 8] \to R$, such that

$$\int_1^2 f^2(t^3)dt + 2\int_1^2 f(t^3)dt = \frac{2}{3}\int_1^8 f(t)dt - \int_1^2 (t^2-1)^2 dt.$$

Solution. Using the substitution $t = u^3$ we get

$$\frac{2}{3}\int_1^8 f(t)dt = 2\int_1^2 u^2 f(u^3)du = 2\int_1^2 t^2 f(t^3)du.$$

Hence, by the assumptions,

$$\int_1^2 \left(f^2(t^3) + (t^2-1)^2 + 2f(t^3) - 2t^2 f(t^3)\right) dt = 0.$$

Since $f^2(t^3)+(t^2-1)^2+2f(t^3)-2t^2f(t^3) = (f(t^3))^2+(1-t^2)^2+2(1-t^2)f(t^3) = \left(f(t^3)+1-t^2\right)^2 \geq 0$, we get

$$\int_1^2 \left(f(t^3)+1-t^2\right)^2 dt = 0.$$

The continuity of f implies that $f(t^3) = t^2 - 1$, $1 \leq t \leq 2$, thus, $f(x) = x^{2/3} - 1$, $1 \leq x \leq 8$.

Remark. If the continuity assumption for f is replaced by Riemann integrability then infinitely many f's would satisfy the given equality. For example if C is any closed nowhere dense and of measure zero subset of $[1, 8]$ (for example a finite set or an appropriate Cantor type set) then any function f such that $f(x) = x^{2/3} - 1$ for every $x \in [1, 8]\backslash C$ satisfies the conditions.

Problem 2

Let $M, N \in M_2(\mathbb{C})$ be two nonzero matrices such that

$$M^2 = N^2 = 0_2 \text{ and } MN + NM = I_2$$

where 0_2 is the 2×2 zero matrix and I_2 the 2×2 unit matrix. Prove that there is an invertible matrix $A \in M_2(\mathbb{C})$ such that

$$M = A\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}A^{-1} \text{ and } N = A\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}A^{-1}.$$

First solution. Consider $f, g : \mathbb{C}^2 \to \mathbb{C}^2$ given by $f(x) = Mx$ and $g(x) = Nx$.
We have $f^2 = g^2 = 0$ and $fg + gf = \mathrm{id}_{\mathbb{C}^2}$; composing the last relation (to the left, for instance) with fg we find that $(fg)^2 = fg$, so fg is a projection of $\mathbb{C}^2$.
If fg were zero, then $gf = \mathrm{id}_{\mathbb{C}^2}$, so f and g would be invertible, thus contradicting $f^2 = 0$.
Therefore, fg is nonzero. Let $u \in \mathrm{Im}(fg) \setminus \{0\}$ and $w \in \mathbb{C}^2$ such that $u = fg(w)$. We obtain $fg(u) = (fg)^2(w) = fg(w) = u$. Let $v = g(u)$. The vector v is nonzero, because otherwise we obtain $u = f(v) = 0$.
Moreover, u and v are not collinear since $v = \lambda u$ with $\lambda \in \mathbb{C}$ implies $u = f(v) = f(\lambda u) = \lambda f(u) = \lambda f^2(g(w)) = 0$, a contradiction.
Let us now consider the ordered basis $\mathcal{B}$ of $\mathbb{C}^2$ consisting of u and v.
We have $f(u) = f^2(g(u)) = 0$, $f(v) = f(g(u)) = u$, $g(u) = v$ and $g(v) = g^2(u) = 0$.
Therefore, the matrices of f and g with respect to $\mathcal{B}$ are $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, respectively.
We take A to be the change of base matrix from the standard basis of $\mathbb{C}^2$ to $\mathcal{B}$ and we are done. $\square$

Second solution. Let us denote $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ by E_{12} and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ by E_{21}. Since $M^2 = N^2 = 0_2$ and $M, N \neq 0_2$, the minimal polynomials of both M and N are equal to x^2. Therefore, there are invertible matrices $B, C \in \mathcal{M}_2(\mathbb{C})$ such that $M = BE_{12}B^{-1}$ and $N = CE_{21}C^{-1}$.
Note that B and C are not uniquely determined. If $B_1E_{12}B_1^{-1} = B_2E_{12}B_2^{-1}$, then $(B_1^{-1}B_2)E_{12} = E_{12}(B_1^{-1}B_2)$; putting $B_1^{-1}B_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the last relation is equivalent to $\begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}$. Consequently, $B_1E_{12}B_1^{-1} = B_2E_{12}B_2^{-1}$ if and only if there exist $a \in \mathbb{C} - \{0\}$ and $b \in \mathbb{C}$ such that

$$B_2 = B_1 \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}. \quad (*)$$

Similarly, $C_1E_{21}C_1^{-1} = C_2E_{21}C_2^{-1}$ if and only if there exist $\alpha \in \mathbb{C} - \{0\}$ and $\beta \in \mathbb{C}$ such that

$$C_2 = C_1 \begin{pmatrix} \alpha & 0 \\ \beta & \alpha \end{pmatrix}. \quad (**)$$

Now, $MN + NM = I_2$, $M = BE_{12}B^{-1}$ and $N = CE_{21}C^{-1}$ give

$$BE_{12}B^{-1}CE_{21}C^{-1} + CE_{21}C^{-1}BE_{12}B^{-1} = I_2,$$

or

$$E_{12}B^{-1}CE_{21}C^{-1}B + B^{-1}CE_{21}C^{-1}BE_{12} = I_2.$$

If $B^{-1}C = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$, the previous relation means

$$\begin{pmatrix} z & t \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ t & -y \end{pmatrix} + \begin{pmatrix} y & 0 \\ t & 0 \end{pmatrix}\begin{pmatrix} 0 & t \\ 0 & -z \end{pmatrix} = (xt - yz)I_2 \neq 0_2.$$

After computations we find this to be equivalent to $xt - yz = t^2 \neq 0$. Consequently, there are $y, z \in \mathbb{C}$ and $t \in \mathbb{C} - \{0\}$ such that

$$C = B \begin{pmatrix} t + \frac{yz}{t} & y \\ z & t \end{pmatrix}. \quad (***)$$

According to $(*)$ and $(**)$, our problem is equivalent to finding $a, \alpha \in \mathbb{C} - \{0\}$ and $b, \beta \in \mathbb{C}$ such that $C \begin{pmatrix} \alpha & 0 \\ \beta & \alpha \end{pmatrix} = B \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$. Taking relation $(***)$ into account, we need to find $a, \alpha \in \mathbb{C} - \{0\}$ and $b, \beta \in \mathbb{C}$ such that

$$B \begin{pmatrix} t + \frac{yz}{t} & y \\ z & t \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ \beta & \alpha \end{pmatrix} = B \begin{pmatrix} a & b \\ 0 & a \end{pmatrix},$$

or, B being invertible,

$$\begin{pmatrix} t + \frac{yz}{t} & y \\ z & t \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ \beta & \alpha \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.$$

This means $\begin{cases} \alpha t + \alpha \dfrac{yz}{t} + \beta y = a \\ \alpha y = b \\ \alpha z + \beta t = 0 \\ \alpha t = a \end{cases}$,

and these conditions are equivalent to $\begin{cases} \alpha y = b \\ \alpha z = -\beta t \\ \alpha t = a \end{cases}$.

It is now enough to choose $\alpha = 1$, $a = t$, $b = y$ and $\beta = -\dfrac{z}{t}$. $\square$

Third Solution. Let f, g be as in the first solution. Since $f^2 = 0$ there exists a nonzero $v_1 \in \mathrm{Ker} f$ so $f(v_1) = 0$ and setting $v_2 = g(v_1)$ we get

$$f(v_2) = (fg + gf)(v_1) = v_1 \neq 0$$

by the assumptions (and so $v_2 \neq 0$). Also

$$g(v_2) = g^2(v_1) = 0$$

and so to complete the proof it suffices to show that v_1 and v_2 are linearly independent, because then the matrices of f, g with respect to the ordered basis (v_1, v_2) would be E_{12} and E_{21} respectively, according to the above relations. But if $v_2 = \lambda v_1$ then $0 = g(v_2) = \lambda g(v_1) = \lambda v_2$ so since $v_2 \neq 0$, λ must be 0 which gives $v_2 = 0v_1 = 0$ contradiction. This completes the proof. $\square$

Remark. A nonelementary solution of this problem can be given by observing that the conditions on M, N imply that the correspondence $I_2 \to I_2, M \to E_{12}$ and $N \to E_{21}$ extends to an isomorphism between the subalgebras of $\mathcal{M}_2(\mathbb{C})$ generated by I_2, M, N and I_2, E_{12}, E_{21} respectively, and then one can apply Noether-Skolem Theorem to show that this isomorphism is actually conjugation by an $A \in Gl_2(\mathbb{C})$ etc.

Problem 3

Find the maximum value of

$$\int_0^1 |f'(x)|^2\, |f(x)|\, \frac{1}{\sqrt{x}}\, dx$$

over all continuously differentiable functions $f : [0,1] \to \mathbb{R}$ *with* $f(0) = 0$ *and*

$$\int_0^1 |f'(x)|^2\, dx \leq 1. \qquad (*)$$

Solution. For $x \in [0,1]$ let

$$g(x) = \int_0^x |f'(t)|^2\, dt.$$

Then for $x \in [0,1]$ the Cauchy-Schwarz inequality gives

$$|f(x)| \leq \int_0^x |f'(t)|\, dt \leq \left(\int_0^x |f'(t)|^2\, dt \right)^{1/2} \sqrt{x} = \sqrt{x} g(x)^{1/2}.$$

Thus

$$\int_0^1 |f'(x)|^2 |f(x)| \frac{1}{\sqrt{x}}\, dx \leq \int_0^1 g(x)^{1/2} g'(x) dx = \frac{2}{3}[g(1)^{3/2} - g(0)^{3/2}]$$
$$= \frac{2}{3} \left(\int_0^1 |f'(t)|^2\, dt \right)^{3/2} \leq \frac{2}{3}.$$

by $(*)$. The maximum is achieved by the function $f(x) = x$. $\square$

Remark. If the condition $(*)$ is replaced by $\int_0^1 |f'(x)|^p dx \leq 1$ with $0 < p < 2$ fixed, then the given expression would have supremum equal to $+\infty$, as it can be seen by considering continuously differentiable functions that approximate the functions $f_M(x) = Mx$ for $0 \leq x \leq \dfrac{1}{M^p}$ and $\dfrac{1}{M^{p-1}}$ for $\dfrac{1}{M^p} < x \leq 1$, where M can be an arbitrary large positive real number.

Problem 4

Let $A \in M_2(Q)$ *such that there is* $n \in N$, $n \neq 0$, *with* $A^n = -I_2$. *Prove that either* $A^2 = -I_2$ *or* $A^3 = -I_2$.

First Solution. Let $f_A(x) = det(A - xI_2) = x^2 - sx + p \in \mathbb{Q}[x]$ be the characteristic polynomial of A and let λ_1, λ_2 be its roots, also known as the eigenvalues of matrix A. We have that $\lambda_1 + \lambda_2 = s \in \mathbb{Q}$ and $\lambda_1\lambda_2 = p \in \mathbb{Q}$. We know, based on Cayley-Hamilton theorem, that the matrix A satisfies the relation $A^2 - sA + pI_2 = 0_2$. For any eigenvalue $\lambda \in \mathbb{C}$ there is an eigenvector $X \neq 0$, such that $AX = \lambda X$. By induction we have that $A^n X = \lambda^n X$ and it follows that $\lambda^n = -1$. Thus, the eigenvalues of A satisfy the equalities

$$\lambda_1^n = \lambda_2^n = -1 \qquad (*).$$

Is $\lambda_1 \in \mathbb{R}$ then we also have that $\lambda_2 \in \mathbb{R}$ and from $(*)$ we get that $\lambda_1 = \lambda_2 = -1$ (and note that n must be odd) so A satisfies the equation $(A + I_2)^2 = A^2 + 2A + I_2 = 0_2$ and it follows that $-I_2 = A^n = (A + I_2 - I_2)^n = n(A + I_2) - I_2$ which gives $A = -I_2$ and hence $A^3 = -I_2$.

If $\lambda_1 \in \mathbb{C} \setminus \mathbb{R}$ then $\lambda_2 = \overline{\lambda_1} \in \mathbb{C} \setminus \mathbb{R}$ and since $\lambda_1^n = -1$ we get that $|\lambda_{1,2}| = 1$ and this implies that $\lambda_{1,2} = \cos t \pm i \sin t$. Now we have the equalities $\lambda_1 + \lambda_2 = 2\cos t = s \in \mathbb{Q}$ and $\lambda_1^n = -1$ implies that $\cos nt + i \sin nt = -1$ which in turn implies that $\cos nt = -1$. Using the equality $\cos(n+1)t + \cos(n-1)t = 2\cos t \cos nt$ we get that there is a polynomial $P_n = x^n + \cdots$ of degree n with integer coefficients such that $2\cos nt = P_n(2\cos t)$. Set $x = 2\cos t$ and observe that we have $P_n(x) = -2$ so $x = 2\cos t$ is a rational root of an equation of the form $x^n + \cdots = 0$. However, the rational roots of this equation are integers, so $x \in \mathbb{Z}$ and since $|x| \leq 2$ we get that $2\cos t = -2, -1, 0, 1, 2$.

When $2\cos t = \pm 2$ then $\lambda_{1,2}$ are real numbers (note that in this case $\lambda_1 = \lambda_2 = 1$ or $\lambda_1 = \lambda_2 = -1$) and this case was discussed above.

When $2\cos t = 0$ we get that $A^2 + I_2 = 0_2$ so $A^2 = -I_2$.

When $2\cos t = 1$ we get that $A^2 - A + I_2 = 0_2$ which implies that $(A + I_2)(A^2 - A + I_2) = 0_2$ so $A^3 = -I_2$.

When $2\cos t = -1$ we get that $A^2 + A + I_2 = 0_2$ and this implies that $(A - I_2)(A^2 + A + I_2) = 0_2$ so $A^3 = I_2$. It follows that $A^n \in \{I_2, A, A^2\}$. However, $A^n = -I_2$ and this implies that either $A = -I_2$ or $A^2 = -I_2$ both of which contradict the equality $A^3 = I_2$. This completes the proof. $\square$

Remark. The polynomials P_n used in the above proof are related to the Chebyshev polynomials, $T_n(x) = \cos(n \arccos x)$. One could also get the conclusion that $2\cos t$ is an integer by considering the sequence $x_m = 2\cos(2^m t)$ and noticing that since $x_{m+1} = x_m^2 - 2$, if x_0 were a noninteger rational $\dfrac{a}{b}$ $(b > 1)$ in lowest terms then the denominator of x_m *in lowest terms* would be b^{2^m} and this contradicts the fact that x_m must be periodic since t is a rational multiple of π.

Second Solution. Let $m_A(x)$ be the minimal polynomial of A. Since $A^{2n} - I_2 = (A^n + I_2)(A^n - I_2) = 0_2$, $m_A(x)$ must be a divisor of $x^{2n} - 1$ which has no multiple roots. It is well known that the monic irreducible over $\mathbb{Q}$ factors of $x^{2n} - 1$ are exactly the cyclotomic polynomials $\Phi_d(x)$ where d divides $2n$. Hence the irreducible over $\mathbb{Q}$ factors of $m_A(x)$ must be cyclotomic polynomials and since the degree of $m_A(x)$ is at most 2 we conclude that $m_A(x)$ itself must be a cyclotomic polynomial, say $\Phi_d(x)$ for some positive integer d with $\phi(d) = 1$ or 2 (where ϕ is the Euler totient function), $\phi(d)$ being the degree of $\Phi_d(x)$. But this implies that $d \in \{1, 2, 3, 4, 6\}$ and since A, A^3 cannot be equal to I_2 we get that $m_A(x) \in \{x + 1, x^2 + 1, x^2 - x + 1\}$ and this implies that either $A^2 = -I_2$ or $A^3 = -I_2$. $\square$