

South Eastern European Mathematical Olympiad for University Students

Plovdiv, Bulgaria

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Problem 1. Let $f_0 : [0, 1] \rightarrow \mathbb{R}$ be a continuous function. Define the sequence of functions $f_n : [0, 1] \rightarrow \mathbb{R}$ by

$$f_n(x) = \int_0^x f_{n-1}(t) dt$$

for all integers $n \geq 1$.

- a) Prove that the series $\sum_{n=1}^{\infty} f_n(x)$ is convergent for every $x \in [0, 1]$.
- b) Find an explicit formula for the sum of the series $\sum_{n=1}^{\infty} f_n(x)$, $x \in [0, 1]$.

Solution 1. a) Clearly $f'_n = f_{n-1}$ for all $n \in \mathbb{N}$. The function f_0 is bounded, so there exists a real positive number M such that $|f_0(x)| \leq M$ for every $x \in [0, 1]$. Then

$$|f_1(x)| \leq \int_0^x |f_0(t)| dt \leq Mx, \quad \forall x \in [0, 1],$$

$$|f_2(x)| \leq \int_0^x |f_1(t)| dt \leq M \frac{x^2}{2}, \quad \forall x \in [0, 1].$$

By induction, it is easy to see that

$$|f_n(x)| \leq M \frac{x^n}{n!}, \quad \forall x \in [0, 1], \forall n \in \mathbb{N}.$$

Therefore

$$\max_{x \in [0, 1]} |f_n(x)| \leq \frac{M}{n!}, \quad \forall n \in \mathbb{N}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n!}$ is convergent, so the series $\sum_{n=1}^{\infty} f_n$ is uniformly convergent on $[0, 1]$.

b) Denote by $F : [0, 1] \rightarrow \mathbb{R}$ the sum of the series $\sum_{n=1}^{\infty} f_n$. The series of the derivatives $\sum_{n=1}^{\infty} f'_n$ is uniformly convergent on $[0, 1]$, since

$$\sum_{n=1}^{\infty} f'_n = \sum_{n=0}^{\infty} f_n$$

and the last series is uniformly convergent. Then the series $\sum_{n=1}^{\infty} f_n$ can be differentiated term by term and $F' = F + f_0$. By solving this equation, we find $F(x) = e^x \left(\int_0^x f_0(t) e^{-t} dt \right)$, $x \in [0, 1]$.

Solution 2. We write

$$\begin{aligned}
f_n(x) &= \int_0^x dt \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-2}} f_0(t_{n-1}) dt_{n-1} \\
&= \int_{0 \leq t_{n-1} \leq \dots \leq t_1 \leq t \leq x} f_0(t_{n-1}) dt dt_1 \dots dt_{n-1} \\
&= \int_{0 \leq t \leq t_1 \leq \dots \leq t_{n-1} \leq x} f_0(t) dt dt_1 \dots dt_{n-1} \\
&= \int_0^x f_0(t) dt \int_t^x dt_1 \int_{t_1}^x dt_2 \dots \int_{t_{n-3}}^x dt_{n-2} \int_{t_{n-2}}^x dt_{n-1} \\
&= \int_0^x f_0(t) \frac{(x-t)^{n-1}}{(n-1)!} dt.
\end{aligned}$$

Thus

$$\sum_{n=1}^N f_n(x) = \int_0^x f_0(t) \left(\sum_{n=1}^N \frac{(x-t)^{n-1}}{(n-1)!} \right) dt.$$

We have

$$\begin{aligned}
e^{x-t} &= \sum_{n=0}^{N-1} \frac{(x-t)^n}{n!} + e^\theta \frac{(x-t)^N}{N!}, \quad \theta \in (0, x-t), \\
\sum_{n=0}^{N-1} \frac{(x-t)^n}{n!} &\rightarrow e^{x-t}, \quad N \rightarrow \infty.
\end{aligned}$$

Hence

$$\begin{aligned}
\left| \int_0^x f_0(t) \left(\sum_{n=0}^{N-1} \frac{(x-t)^n}{n!} \right) dt - \int_0^x f_0(t) e^{x-t} dt \right| &\leq \int_0^x |f_0(t)| e^{x-t} \frac{(x-t)^N}{N!} dt \\
&\leq \frac{1}{N!} \int_0^x |f_0(t)| e^{x-t} dt \rightarrow 0, \quad N \rightarrow \infty.
\end{aligned}$$

Problem 2. Inside a square consider circles such that the sum of their circumferences is twice the perimeter of the square.

- Find the minimum number of circles having this property.
- Prove that there exist infinitely many lines which intersect at least 3 of these circles.

Solution. a) Consider the circles C_1, C_2, \dots, C_k with diameters d_1, d_2, \dots, d_k , respectively. Denote by s the length of the square side. By using the hypothesis, we get

$$\pi(d_1 + d_2 + \dots + d_k) = 8s.$$

Since $d_i \leq s$ for $i = 1, \dots, k$, we have

$$8s = \pi(d_1 + d_2 + \dots + d_k) \leq \pi ks,$$

which implies $k \geq \frac{8}{\pi} \cong 2.54$. Hence, there are at least 3 circles inside the square.

b) Project the circles onto one side of the square so that their images are their diameters. Since the sum of the diameters is approximately $2.54s$ and there are at least three circles in the

square, there exists an interval where at least three diameters are overlapping. The lines, passing through this interval and perpendicular to the side on which the diameters are projected, are the required lines.

Problem 3. Denote by $\mathcal{M}_2(\mathbb{R})$ the set of all 2×2 matrices with real entries. Prove that:

- a) for every $A \in \mathcal{M}_2(\mathbb{R})$ there exist $B, C \in \mathcal{M}_2(\mathbb{R})$ such that $A = B^2 + C^2$;
- b) there do not exist $B, C \in \mathcal{M}_2(\mathbb{R})$ such that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = B^2 + C^2$ and $BC = CB$.

Solution. a) Recall that every 2×2 matrix A satisfies $A^2 - (\text{tr} A) A + (\det A) E = 0$. It is clear that

$$\lim_{t \rightarrow +\infty} \text{tr}(A + tE) = +\infty \quad \text{and} \quad \lim_{t \rightarrow +\infty} \frac{\det(A + tE)}{\text{tr}(A + tE)} - t = \lim_{t \rightarrow +\infty} \frac{\det A - t^2}{\text{tr}(A + tE)} = -\infty.$$

Thus, for t large enough one has

$$\begin{aligned} A &= (A + tE) - tE = \frac{1}{\text{tr}(A + tE)} (A + tE)^2 + \left(\frac{\det(A + tE)}{\text{tr}(A + tE)} - t \right) E \\ &= \left(\frac{1}{\sqrt{\text{tr}(A + tE)}} (A + tE) \right)^2 + \left(\sqrt{t - \frac{\det(A + tE)}{\text{tr}(A + tE)}} \right)^2 (-E) \\ &= \left(\frac{1}{\sqrt{\text{tr}(A + tE)}} (A + tE) \right)^2 + \left(\sqrt{t - \frac{\det(A + tE)}{\text{tr}(A + tE)}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)^2. \end{aligned}$$

b) No. For $B, C \in \mathcal{M}_2(\mathbb{R})$, consider $B + iC, B - iC \in \mathcal{M}_2(\mathbb{C})$. If $BC = CB$ then $(B + iC)(B - iC) = B^2 + C^2$. Thus

$$\det(B^2 + C^2) = \det(B + iC) \det(B - iC) = |B + iC|^2 \geq 0,$$

which contradicts the fact that $\det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$.

Problem 4. Suppose that A and B are $n \times n$ matrices with integer entries, and $\det B \neq 0$. Prove that there exists $m \in \mathbb{N}$ such that the product AB^{-1} can be represented as

$$AB^{-1} = \sum_{k=1}^m N_k^{-1},$$

where N_k are $n \times n$ matrices with integer entries for all $k = 1, \dots, m$, and $N_i \neq N_j$ for $i \neq j$.

Solution. Suppose first that $n = 1$. Then we may consider the integer 1×1 matrices as integer numbers. We shall prove that for given integers p and q we can find integers n_1, \dots, n_m such that $\frac{p}{q} = \frac{1}{n_1} + \frac{1}{n_2} + \dots + \frac{1}{n_m}$ and $n_i \neq n_j$ for $i \neq j$.

In fact this is well known as the ‘‘Egyptian problem’’. We write $\frac{p}{q} = \frac{1}{q} + \frac{1}{q} + \dots + \frac{1}{q}$ (p times) and ensure different denominators in the last sum by using several times the equality $\frac{1}{x} = \frac{1}{x+1} + \frac{1}{x(x+1)}$. For example, $\frac{3}{5} = \frac{1}{5} + \frac{1}{5} + \frac{1}{5}$, where we keep the first fraction, we write $\frac{1}{5} = \frac{1}{6} + \frac{1}{30}$ for the second fraction, and $\frac{1}{5} = \frac{1}{7} + \frac{1}{42} + \frac{1}{31} + \frac{1}{930}$ for the third fraction. Finally,

$$\frac{3}{5} = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{30} + \frac{1}{31} + \frac{1}{42} + \frac{1}{930}.$$

Now consider $n > 1$.

CASE 1. Suppose that A is a nonsingular matrix. Denote by λ the least common multiple of the denominators of the elements of the matrix A^{-1} . Hence the matrix $C = \lambda BA^{-1}$ is integer and nonsingular, and one has

$$AB^{-1} = \lambda C^{-1}.$$

According to the case $n = 1$, we can write

$$\lambda = \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_m},$$

where $n_i \neq n_j$ for $i \neq j$. Then

$$AB^{-1} = (n_1 C)^{-1} + (n_2 C)^{-1} + \cdots + (n_m C)^{-1}.$$

It is easy to see that $n_i C \neq n_j C$ for $i \neq j$.

CASE 2. Now suppose that A is singular. First we will show that

$$A = Y + Z,$$

where Y and Z are nonsingular. If $A = (a_{ij})$, for every $i = 1, 2, \dots, n$ we choose an integer x_i such that $x_i \neq 0$ and $x_i \neq a_{ii}$. Define

$$y_{ij} = \begin{cases} a_{ij}, & \text{if } i < j \\ x_i, & \text{if } i = j \\ 0, & \text{if } i > j \end{cases} \quad \text{and} \quad z_{ij} = \begin{cases} 0, & \text{if } i < j \\ a_{ii} - x_i, & \text{if } i = j \\ a_{ij}, & \text{if } i > j. \end{cases}$$

Clearly, the matrices $Y = (y_{ij})$ and $Z = (z_{ij})$ are nonsingular. Moreover, $A = Y + Z$.

From Case 1 we have

$$YB^{-1} = \sum_{r=1}^k (n_r C)^{-1}, \quad ZB^{-1} = \sum_{q=1}^l (m_q D)^{-1},$$

where

$$YB^{-1} = \lambda C^{-1}, \quad \lambda = \sum_{r=1}^k \frac{1}{n_r} \quad \text{and} \quad ZB^{-1} = \mu D^{-1}, \quad \mu = \sum_{q=1}^l \frac{1}{m_q},$$

C and D are integer and nonsingular. Hence,

$$AB^{-1} = \sum_{r=1}^k (n_r C)^{-1} + \sum_{q=1}^l (m_q D)^{-1}.$$

It remains to show that $n_r C \neq m_q D$ for $r = 1, 2, \dots, k$ and $q = 1, 2, \dots, l$. Indeed, assuming that $n_r C = m_q D$ and recalling that $m_q > 0$ we find $D = \frac{n_r}{m_q} C$. Hence $ZB^{-1} = \mu D^{-1} = \frac{\mu m_q}{n_r} C^{-1}$, and then $AB^{-1} = YB^{-1} + ZB^{-1} = \lambda C^{-1} + \frac{\mu m_q}{n_r} C^{-1} = \left(\lambda + \frac{\mu m_q}{n_r} \right) C^{-1}$. We have $\lambda + \frac{\mu m_q}{n_r} > 0$, and C^{-1} is nonsingular. Then AB^{-1} is nonsingular, and therefore A is nonsingular. This is a contradiction.