SEEMOUS 2008
South Eastern European Mathematical Olympiad
for University Students

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Problem 1

Let $f : [1, \infty) \to (0, \infty)$ be a continuous function. Assume that for every $a > 0$, the equation $f(x) = ax$ has at least one solution in the interval $[1, \infty)$.

(a) Prove that for every $a > 0$, the equation $f(x) = ax$ has infinitely many solutions.

(b) Give an example of a strictly increasing continuous function f with these properties.

Problem 2

Let $P_0, P_1, P_2, \ldots$ be a sequence of convex polygons such that, for each $k \geq 0$, the vertices of P_{k+1} are the midpoints of all sides of P_k. Prove that there exists a unique point lying inside all these polygons.

Problem 3

Let $\mathcal{M}_n(\mathbb{R})$ denote the set of all real $n \times n$ matrices. Find all surjective functions $f : \mathcal{M}_n(\mathbb{R}) \to \{0, 1, \ldots, n\}$ which satisfy

$$f(XY) \leq \min\{f(X), f(Y)\}$$

for all $X, Y \in \mathcal{M}_n(\mathbb{R})$.

Problem 4

Let n be a positive integer and $f : [0, 1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 x^k f(x)\, dx = 1$$

for every $k \in \{0, 1, \ldots, n-1\}$. Prove that

$$\int_0^1 (f(x))^2\, dx \geq n^2.$$

Answers

Problem 1

Solution. (a) Suppose that one can find constants $a > 0$ and $b > 0$ such that $f(x) \neq ax$ for all $x \in [b, \infty)$. Since f is continuous we obtain two possible cases:

1.) $f(x) > ax$ for $x \in [b, \infty)$. Define

$$c = \min_{x \in [1,b]} \frac{f(x)}{x} = \frac{f(x_0)}{x_0}.$$

Then, for every $x \in [1, \infty)$ one should have

$$f(x) > \frac{\min(a, c)}{2} x,$$

a contradiction.

2.) $f(x) < ax$ for $x \in [b, \infty)$. Define

$$C = \max_{x \in [1,b]} \frac{f(x)}{x} = \frac{f(x_0)}{x_0}.$$

Then,

$$f(x) < 2 \max(a, C) x$$

for every $x \in [1, \infty)$ and this is again a contradiction.

(b) Choose a sequence $1 = x_1 < x_2 < \cdots < x_k < \cdots$ such that the sequence $y_k = 2^{k \cos k\pi} x_k$ is also increasing. Next define $f(x_k) = y_k$ and extend f linearly on each interval $[x_{k-1}, x_k]$: $f(x) = a_k x + b_k$ for suitable a_k, b_k. In this way we obtain an increasing continuous function f, for which $\lim\limits_{n\to\infty} \frac{f(x_{2n})}{x_{2n}} = \infty$ and $\lim\limits_{n\to\infty} \frac{f(x_{2n-1})}{x_{2n-1}} = 0$. It now follows that the continuous function $\frac{f(x)}{x}$ takes every positive value on $[1, \infty)$.

Problem 2

Solution. For each $k \geq 0$ we denote by $A_i^k = (x_i^k, y_i^k)$, $i = 1, \ldots, n$ the vertices of P_k. We may assume that the center of gravity of P_0 is $O = (0, 0)$; in other words,

$$\frac{1}{n}(x_1^0 + \cdots + x_n^0) = 0 \ \text{ and } \ \frac{1}{n}(y_1^0 + \cdots + y_n^0) = 0.$$

Since $2x_i^{k+1} = x_i^k + x_{i+1}^k$ and $2y_i^{k+1} = y_i^k + y_{i+1}^k$ for all k and i (we agree that $x_{n+j}^k = x_j^k$ and $y_{n+j}^k = y_j^k$) we see that

$$\frac{1}{n}(x_1^k + \cdots + x_n^k) = 0 \ \text{ and } \ \frac{1}{n}(y_1^k + \cdots + y_n^k) = 0$$

for all $k \geq 0$. This shows that $O = (0, 0)$ is the center of gravity of all polygons P_k.

In order to prove that O is the unique common point of all P_k's it is enough to prove the following claim:

Claim. Let R_k be the radius of the smallest ball which is centered at O and contains P_k. Then, $\lim\limits_{k\to\infty} R_k = 0$.

Proof of the Claim. Write $\|\cdot\|_2$ for the Euclidean distance to the origin O. One can easily check that there exist $\beta_1, \ldots, \beta_n > 0$ and $\beta_1 + \cdots + \beta_n = 1$ such that

$$A_j^{k+n} = \sum_{i=1}^{n} \beta_i A_{j+i-1}^k$$

for all k and j. Let $\lambda = \min_{i=1,\ldots,n} \beta_i$. Since $O = \sum_{i=1}^n A_{j+i-1}^k$, we have the following:

$$\begin{aligned}
\|A_j^{k+n}\|_2 &= \left\| \sum_{i=1}^{n} (\beta_i - \lambda) A_{j+i-1}^k \right\|_2 \\
&\leq \sum_{i=1}^{n} (\beta_i - \lambda) \|A_{j+i-1}^k\|_2 \\
&\leq R_k \sum_{i=1}^{n} (\beta_i - \lambda) = R_k(1 - n\lambda).
\end{aligned}$$

This means that P_{k+n} lies in the ball of radius $R_k(1-n\lambda)$ centered at O. Observe that $1 - n\lambda < 1$.

Continuing in the same way we see that P_{mn} lies in the ball of radius $R_0(1-n\lambda)^m$ centered at O. Therefore, $R_{mn} \to 0$. Since $\{R_n\}$ is decreasing, the proof is complete.

Problem 3

Solution. We will show that the only such function is $f(X) = \operatorname{rank}(X)$. Setting $Y = I_n$ we find that $f(X) \leq f(I_n)$ for all $X \in \mathcal{M}_n(\mathbb{R})$. Setting $Y = X^{-1}$ we find that $f(I_n) \leq f(X)$ for all invertible $X \in \mathcal{M}_n(\mathbb{R})$. From these facts we conclude that $f(X) = f(I_n)$ for all $X \in GL_n(\mathbb{R})$.

For $X \in GL_n(\mathbb{R})$ and $Y \in \mathcal{M}_n(\mathbb{R})$ we have

$$\begin{aligned}
&f(Y) = f(X^{-1}XY) \leq f(XY) \leq f(Y), \\
&f(Y) = f(YXX^{-1}) \leq f(YX) \leq f(Y).
\end{aligned}$$

Hence we have $f(XY) = f(YX) = f(Y)$ for all $X \in GL_n(\mathbb{R})$ and $Y \in \mathcal{M}_n(\mathbb{R})$. For $k = 0, 1, \ldots, n$, let

$$J_k = \begin{pmatrix} I_k & O \\ O & O \end{pmatrix}.$$

It is well known that every matrix $Y \in \mathcal{M}_n(\mathbb{R})$ is equivalent to J_k for $k = \operatorname{rank}(Y)$. This means that there exist matrices $X, Z \in GL_n(\mathbb{R})$ such that $Y = XJ_kZ$. From the discussion above it follows that $f(Y) = f(J_k)$. Thus it suffices to determine the values of the function f on the matrices $J_0, J_1, \ldots, J_n$. Since $J_k = J_k \cdot J_{k+1}$ we have $f(J_k) \leq f(J_{k+1})$ for $0 \leq k \leq n-1$. Surjectivity of f imples that $f(J_k) = k$ for $k = 0, 1, \ldots, n$ and hence $f(Y) = \operatorname{rank}(Y)$ for all $Y \in \mathcal{M}_n(\mathbb{R})$.

Problem 4

Solution. There exists a polynomial $p(x) = a_1 + a_2x + \cdots + a_nx^{n-1}$ which satisfies

$$\int_0^1 x^k p(x)\, dx = 1 \quad \text{for all } k = 0, 1, \ldots, n-1. \tag{1}$$

It follows that, for all $k = 0, 1, \ldots, n-1$,

$$\int_0^1 x^k (f(x) - p(x))\, dx = 0,$$

and hence

$$\int_0^1 p(x)(f(x) - p(x))\, dx = 0.$$

Then, we can write

$$\begin{aligned}
\int_0^1 (f(x) - p(x))^2 dx &= \int_0^1 f(x)(f(x) - p(x))\, dx \\
&= \int_0^1 f^2(x)\, dx - \sum_{k=0}^{n-1} a_{k+1} \int_0^1 x^k f(x)\, dx,
\end{aligned}$$

and since the first integral is non-negative we get

$$\int_0^1 f^2(x)\, dx \geq a_1 + a_2 + \cdots + a_n.$$

To complete the proof we show the following:

Claim. For the coefficients $a_1, \ldots, a_n$ of p we have

$$a_1 + a_2 + \cdots + a_n = n^2.$$

Proof of the Claim. The defining property of p can be written in the form

$$\frac{a_1}{k+1} + \frac{a_2}{k+2} + \cdots + \frac{a_n}{k+n} = 1, \qquad 0 \leq k \leq n-1.$$

Equivalently, the function

$$r(x) = \frac{a_1}{x+1} + \frac{a_2}{x+2} + \cdots + \frac{a_n}{x+n} - 1$$

has $0, 1, \ldots, n-1$ as zeros. We write r in the form

$$r(x) = \frac{q(x) - (x+1)(x+2)\cdots(x+n)}{(x+1)(x+2)\cdots(x+n)},$$

where q is a polynomial of degree $n-1$. Observe that the coefficient of x^{n-1} in q is equal to $a_1 + a_2 + \cdots + a_n$. Also, the numerator has $0, 1, \ldots, n-1$ as zeros, and since $\lim_{x\to\infty} r(x) = -1$ we must have

$$q(x) = (x+1)(x+2)\cdots(x+n) - x(x-1)\cdots(x-(n-1)).$$

This expression for q shows that the coefficient of x^{n-1} in q is $\frac{n(n+1)}{2} + \frac{(n-1)n}{2}$. It follows that

$$a_1 + a_2 + \cdots + a_n = n^2.$$